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# REDUCTION OF SOCIAL CHOICE PROBLEMS: A SIMPLE PROOF OF ARROW'S GENERAL POSSIBILITY THEOREM

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## I. Introduction

In this note, we show that there exists an Arrovian social welfare function for a society with  $n-1$  members ( $n \geq 3$ ) if there exists an Arrovian social welfare function for a society with  $n$  members. By the repeated use of this property, we are assured of the existence of an Arrovian social welfare function for a two-person society. It is easy to verify, however, that such a function does not exist for a two-person society, which annihilates an Arrovian social welfare function for a society of any finite size by one stroke. It is hoped that this way of viewing social choice problems might be of some help, especially from the pedagogical point of view.<sup>1</sup>

## II. Social Welfare Function

Consider an  $n$ -person society ( $n \geq 2$ ). Let  $I(n) := \{1, 2, \dots, n\}$  be the set of all individuals.  $X$  denotes the set of all social alternatives, which is assumed to contain at least three elements.

Each individual  $i \in I(n)$  has a complete and transitive preference ordering  $R_i$  on  $X$ . The corresponding strict preference ordering will be denoted by  $P_i$ .<sup>2</sup> An  $n$ -list of preference orderings, one ordering for each individual, will be called a *profile*, to be denoted by  $a = (R_1^a, R_2^a, \dots, R_n^a)$ . A *social welfare function* (SWF) for an  $n$ -person society is a function  $F_n$  which sends each profile  $a = (R_1^a, R_2^a, \dots, R_n^a)$  into a social preference ordering  $R^a = F_n(a)$ . Throughout this note, we assume that the domain,  $d(F_n)$ , of  $F_n$  consists of all logically possible profiles. To be formal, let  $A$  denote the set of all complete and transitive orderings on  $X$  and let  $A^{(n)}$  be the  $n$ -fold Cartesian product of  $A$ . Then  $d(F_n) = A^{(n)}$  and  $r(F_n) \subset A$ , where  $r(F_n)$  is the range of  $F_n$ .

An SWF is called *Arrovian* if it satisfies all of the following three requirements.

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<sup>1</sup> The current standard of the proof of Arrow's theorem may be found in Barberà (1983), Fishburn (1970) and Sen (1979).

<sup>2</sup> Formally speaking,  $xP_iy$  holds if and only if  $(xR_iy \ \& \ \text{not } yR_ix)$  holds.

*Pareto:* For every profile  $a \in d(F_n)$  and  $x, y \in X$ , if  $xP_i^a y$  for all  $i \in I(n)$ , then  $xP^a y$ , where  $P^a$  is the strict social preference corresponding to  $R^a = F_n(a)$ .

*Independence of Irrelevant Alternatives:* Suppose that two profiles  $a = (R_1^a, R_2^a, \dots, R_n^a) \in d(F_n)$  and  $b = (R_1^b, R_2^b, \dots, R_n^b) \in d(F_n)$  coincide on  $\{x, y\}$ , where  $x, y \in X$ . Then  $R^a = F_n(a)$  and  $R^b = F_n(b)$  also coincide on  $\{x, y\}$ .

*Non-Dictatorship:* There exists no dictator for  $F_n$ , where a *dictator* for  $F_n$  is an individual  $d \in I(n)$  such that, for every profile  $a \in d(F_n)$  and  $x, y \in X$ ,  $xP_d^a y$  implies  $xP^a y$ , where  $P^a$  is the strict social preference corresponding to  $R^a = F_n(a)$ .

### III. Arrow's General Possibility Theorem

Let us start with two auxiliary lemmas which prove useful later on.

*Dictator Lemma:* Let  $F_n$  be an SWF satisfying Pareto and Independence axioms, where  $n \geq 2$ . If there exist  $i \in I(n)$ ,  $a \in d(F_n)$  and  $x, y \in X$  such that

$$xP_i^a y \text{ \& } (\forall j \in I(n) - \{i\} : yP_j^a x) \rightarrow xP^a y, \quad (1)$$

then  $i$  is a dictator for  $F_n$ .

*Proof.* Take any  $z \in X - \{x, y\}$  and let  $b \in d(F_n)$  be such that  $i$  prefers  $x$  to  $y$  to  $z$ , and all others prefer  $y$  to  $x$  and  $y$  to  $z$ . No specification is made of  $R_j^b (j \neq i)$  over  $\{x, z\}$ . By Independence and (1), we have  $xP^b y$ , whereas Pareto implies  $yP^b z$ , so that we obtain  $xP^b z$  by transitivity of  $R^b = F_n(b)$ .

Take any  $v, w \in X$  with  $v \neq w$ , and consider the case where  $\{x, z\} \cap \{v, w\} = \emptyset$ . Let  $c \in d(F_n)$  be such that  $i$  prefers  $v$  to  $x$  to  $z$  to  $w$ , whereas all others prefer  $z$  to  $w$  and  $v$  to  $x$ . No specification is made of  $R_j^c (j \neq i)$  over  $\{x, z\}$  and  $\{v, w\}$ . By Independence we obtain  $xP^c z$ , whereas Pareto entails  $vP^c x$  and  $zP^c w$ . By transitivity of  $R^c = F_n(c)$ , we then have  $vP^c w$ . By Independence this implies that  $i$  can get his way over  $\{v, w\}$ .

The other cases, where  $\{x, z\} \cap \{v, w\} \neq \emptyset$ , can be treated similarly. ||

*Reduction Lemma:* If an Arrovian social welfare function  $F_n$  exists, where  $n \geq 3$ , then there exists an Arrovian social welfare function  $F_{n-1}$ .

*Proof.* Let  $R_n^0$  be the universal binary relation on  $X$ , viz.,  $R_n^0 := X \times X$ , and define, for every  $(R_1, R_2, \dots, R_{n-1}) \in A^{(n-1)}$ :

$$F_{n-1}(R_1, R_2, \dots, R_{n-1}) := F_n(R_1, R_2, \dots, R_{n-1}, R_n^0). \quad (2)$$

We show that  $F_{n-1}$  is indeed an Arrovian social welfare function. It is clear that  $d(F_{n-1}) = A^{(n-1)}$  and that  $F_{n-1}$  inherits Independence property from that of  $F_n$ .

To show Non-Dictatorship property for  $F_{n-1}$ , suppose to the contrary that  $d \in I(n-1)$  is a dictator for  $F_{n-1}$ . Let  $x, y \in X$  and  $a \in d(F_{n-1})$  be such that  $xP_d^a y$  and  $yP_j^a x$  for all  $j \in I(n-1) - \{d\}$ , whereas  $yI_n^0 x$  by construction. Since  $d$  is a dictator for  $F_{n-1}$ ,  $xP^a y$  holds, where  $P^a$  is the asymmetric part of  $R^a = F_{n-1}(a)$ . Take any  $z \in X - \{x, y\}$  and let  $b \in d(F_n)$  be such that  $d$  prefers  $x$  to  $y$  to  $z$ ,  $n$  is indifferent between  $x$  and  $y$  and prefers each to  $z$ , and everyone in  $I(n) - \{d, n\}$  prefers  $y$  to  $z$  to  $x$ . By Independence  $xP^b y$ , and by Pareto  $yP^b z$ ,

which imply  $xP^bz$  by transitivity of  $R^b=F_n(b)$ . Take any  $w \in X - \{x, y\}$  and let  $c \in d(F_n)$  be such that  $d$  prefers  $x$  to  $z$  to  $w$ ,  $n$  prefers  $w$  to  $x$  to  $z$ , and everyone in  $I(n) - \{d, n\}$  prefers  $z$  to  $w$  to  $x$ . By Independence  $xP^cz$ , whereas by completeness of  $R^c=F_n(c)$ , either  $xP^cw$  or  $wR^cx$  must be the case. In the former case,  $d$  is a dictator for  $F_n$ , whereas in the latter case, we get  $wP^cz$  by transitivity and  $n$  turns out to be a dictator for  $F_n$ . Either case, we obtain a contradiction.

To show Pareto property for  $F_{n-1}$ , let  $x, y \in X$  and  $a \in d(F_{n-1})$  be such that  $xP_i^ay$  for all  $i \in I(n-1)$ . By construction, we have  $xI_n^0y$ . Suppose that  $yR^ax$  were true, where  $R^a=F_{n-1}(a)$ . Take any  $z \in X - \{x, y\}$  and let  $b \in d(F_n)$  be such that everyone in  $I(n-1)$  prefers  $x$  to  $z$  to  $y$ , whereas  $n$  prefers  $z$  to both  $x$  and  $y$ , and is indifferent between  $x$  and  $y$ . By Independence  $yR^bx$ , whereas Pareto on  $F_n$  entails  $zP^by$ , so that  $zP^bx$  by transitivity. Then  $n$  is a dictator for  $F_n$ , a contradiction.  $\parallel$

We are now ready to prove:

*Arrow's General Possibility Theorem:* There exists no Arrovian social welfare function.

*Proof.* If there exists an Arrovian social welfare function  $F_n (n \geq 3)$ , then Reduction Lemma can be used repeatedly to assure the existence of a sequence of the Arrovian social welfare functions  $F_{n-1}, F_{n-2}, \dots, F_2$ . To prove the theorem for any  $n \geq 3$ , we have only to prove that  $F_2$  cannot exist.

Let  $x, y \in X$  and  $a \in d(F_2)$  be such that 1 prefers  $x$  to  $y$  and 2 prefers  $y$  to  $x$ . There are two possibilities to be considered separately, viz., (1)  $xR^ay$ , and (2)  $yP^ax$ , where  $R^a=F_2(a)$ . In the case (2), 2 is a dictator for  $F_2$  by Dictator Lemma. Suppose, therefore, that (1) is the case, and let  $z \in X - \{x, y\}$  and  $b \in d(F_2)$  be such that 1 prefers  $x$  to  $y$  to  $z$ , and 2 prefers  $y$  to  $z$  to  $x$ . By Independence  $xR^by$ , and by Pareto  $yP^bz$ , which entail  $xP^bz$ , where  $R^b=F_2(b)$ . Therefore 1 turns out to be a dictator for  $F_2$  by virtue of Dictator Lemma. This completes our proof.  $\parallel$

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